

# A Liouville theorem for subcritical Lane-Emden system

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## Abstract

In this paper, we present a necessary and sufficient condition to the Lane-Emden conjecture. This condition is an energy type of integral estimate on solutions to subcritical Lane-Emden system. To approach the long standing and interesting conjecture, we believe that one plausible path is to refocus on establishing this energy type estimate.

## 1 Introduction

This paper is devoted to the nonexistence of positive solution to the Lane-Emden system,

$$\begin{cases} -\Delta u = v^p, \\ -\Delta v = u^q, \end{cases} \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

where  $u, v \geq 0$ ,  $0 < p, q < +\infty$ . The hyperbola [10, 11]

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}$$

is called critical curve because it is known that on or above it, i.e.

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n},$$

which is called critical and supercritical respectively, the system (1.1) admits (radial) non-trivial solutions, cf. Serrin and Zou [18], Liu, Guo and Zhang [9] and Li [8]. However, for subcritical cases, i.e.  $(p, q)$  satisfying,

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}, \quad (1.2)$$

people guess that the following statement holds and call it the Lane-Emden conjecture:

**Conjecture.**  $u = v \equiv 0$  is the unique nonnegative solution for system (1.1).

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The full Lane-Emden conjecture is still open. Only partial results are known, and many researchers have made contribution in pushing the progress forward. We shall briefly present some important recent developments of the Lane-Emden conjecture.

Denote the scaling exponents of system (1.1) by

$$\alpha = \frac{2(p+1)}{pq-1}, \quad \beta = \frac{2(q+1)}{pq-1}, \quad \text{for } pq > 1. \quad (1.3)$$

Then subcritical condition (1.2) is equivalent to

$$\alpha + \beta > n - 2, \quad \text{for } pq > 1. \quad (1.4)$$

For  $p, q$  in the following region

$$pq \leq 1, \text{ or } pq > 1 \text{ and } \max\{\alpha, \beta\} \geq n - 2, \quad (1.5)$$

(1.1) admits no positive entire *supersolution*, cf. Serrin and Zou [17]. This implies the conjecture for  $n = 1, 2$ . Also, the conjecture is true for

$$\min\{\alpha, \beta\} \geq \frac{n-2}{2}, \text{ with } (\alpha, \beta) \neq \left(\frac{n-2}{2}, \frac{n-2}{2}\right), \quad (1.6)$$

cf. Busca and Manásevich [2]. Note that (1.6) covers the case that both  $(p, q)$  are subcritical, i.e.  $\max\{p, q\} \leq \frac{n+2}{n-2}$ , with  $(p, q) \neq \left(\frac{n+2}{n-2}, \frac{n+2}{n-2}\right)$ , which is treated earlier, cf. de Figueiredo and Felmer [4] and Reichel and Zou [16]. Also, Mitidieri [11] has proved that the system admits no radial positive solution. Chen and Li [3] have proved that any solution with finite energy must be radial, therefore combined with Mitidieri [11], no finite-energy non-trivial solution exists.

For  $n = 3$ , the conjecture is solved by two papers. First, Serrin and Zou [17] proved that there is no positive solution with polynomial growth at infinity. Then Poláčik, Quittner and Souplet [14] removed the growth condition. In fact, they proved that no bounded positive solution implies no positive solution. This result has two important consequences. One is that combining with Serrin and Zou's result, one can prove the conjecture for  $n = 3$ . The other is that proving the Lane-Emden conjecture is equivalent to proving nonexistence of bounded positive solution. Thus, we always assume that  $(u, v)$  are bounded in this paper.

For  $n = 4$ , the conjecture is recently solved by Souplet [19]. In [17], Serrin and Zou used the integral estimates to derive the nonexistence results. Souplet further developed the approach of integral estimates and solved the conjecture for  $n = 4$  along the case  $n = 3$ . In higher dimensions, this approach provides a new subregion where the conjecture holds, but the problem of full range in high dimensional space still seems stubborn. Souplet has proved that if

$$\max\{\alpha, \beta\} > n - 3, \quad (1.7)$$

then (1.1) with  $(p, q)$  satisfying (1.2) has no positive solution. Notice that (1.7) covers (1.2) only when  $n \leq 4$ , and when  $n \geq 5$  (1.7) covers a subregion of (1.2).

The approach developed by Souplet in [19] is also effective on non-existence of positive solution to Hardy-Hénon type equations and systems (cf. [5, 6, 12, 13]):

$$\begin{cases} -\Delta u = |x|^a v^p, \\ -\Delta v = |x|^b u^q, \end{cases} \quad \text{in } \mathbb{R}^n.$$

This approach can also be applied to more general elliptic systems, for further details, we refer to [20] and [15]. Moreover, a natural extension and application of this tool is the high order Lane-Emden system which was done by Arthur, Yan and Zhao [1].

In this paper, we point out that the key to the Lane-Emden conjecture is obtaining a certain type of energy estimate. This estimate is in fact a necessary and sufficient condition to the conjecture. Connecting the estimate and the conjecture is a laborious work and needs to combine many types of estimates. We believe that with the result here people can refocus on proving the crucial estimate and thus solve the conjecture.

**Theorem 1.1.** *Let  $n \geq 3$  and  $(u, v)$  be a non-negative bounded solution to (1.1). Assume there exists an  $s > 0$  satisfying  $n - s\beta < 1$  such that*

$$\int_{B_R} v^s \leq CR^{n-s\beta}, \quad (1.8)$$

*then  $u, v \equiv 0$  provided  $0 < p, q < +\infty$  and  $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n}$ .*

**Remark 1.2.** (a) Energy estimate (1.8) is a necessary condition to the Lane-Emden conjecture. One just needs to notice that when  $u, v \equiv 0$ , (1.8) is obviously satisfied. The key to the proof of Theorem 1.1 is to show (1.8) is sufficient.

(b) If  $p \geq q$ , the assumption on  $v$  is weaker than the corresponding assumption on  $u$  due to a comparison principle between  $u$  and  $v$  (i.e. Lemma 2.5).

In other words, if  $p \geq q$ , and we assume for some  $r > 0$ , such that  $n - r\alpha < 1$ ,

$$\int_{B_R} u^r \leq CR^{n-r\alpha}. \quad (1.9)$$

Then (1.9) implies (1.8) by Lemma 2.5.

(c) By taking  $s = p$  Theorem 1.1 recovers the result in [19].

(d) A technical issue is that the standard  $W^{2,p}$ -estimate used in [19] is not enough to establish Theorem 1.1 (see the footnote of Proposition 2.11). To overcome this difficulty, a mixed type  $W^{2,p}$ -estimate is introduced in Lemma 2.2.

**Remark 1.3.** (a) It is worthy to point out an interesting role that the coefficient “1” of the nonlinear source term plays in the Lane-Emden system. Consider the following system

$$\begin{cases} -\Delta u = c_1(x)v^p, \\ -\Delta v = c_2(x)u^q, \end{cases} \quad \text{in } \mathbb{R}^n, \quad (1.10)$$

where  $0 < a \leq c_1(x), c_2(x) \leq b < \infty$  and  $x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) \geq 0$  for some positive constants  $a, b > 0$ . We can also have the following Rellich-Pohožaev type identity for some constants  $d_1, d_2$  such that  $d_1 + d_2 = n - 2$ ,

$$\begin{aligned} & \int_{B_R} \left( \frac{nc_1}{p+1} - d_1c_1 + \frac{x \cdot \nabla c_1(x)}{p+1} \right) v^{p+1} + \left( \frac{nc_2}{q+1} - d_2c_2 + \frac{x \cdot \nabla c_2(x)}{q+1} \right) u^{q+1} dx \\ &= R^n \int_{\mathbb{S}^{n-1}} \frac{c_1(R)v^{p+1}(R)}{p+1} + \frac{c_2(R)u^{q+1}(R)}{q+1} d\sigma \\ &+ R^{n-1} \int_{\mathbb{S}^{n-1}} d_1v'u + d_2u'vd\sigma + R^n \int_{\mathbb{S}^{n-1}} (v'u' - R^{-2}\nabla_\theta u \cdot \nabla_\theta v) d\sigma. \end{aligned} \quad (1.11)$$

By the constraints on  $c_1(x), c_2(x)$ , we can have the left terms (LT) in (1.11) as

$$LT \geq \delta_0 \int_{B_R} v^{p+1} + u^{q+1} dx, \quad \text{for some } \delta_0 > 0. \quad (1.12)$$

The argument in [19] is also valid for this case, and we still can prove nonexistence for  $n \leq 4$  and for  $\max(\alpha, \beta) > n - 3, n \geq 5$ .

On the other hand, for  $c_1(x), c_2(x)$  such that  $x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) < 0$ , there exist non-zero solutions of (1.10) in some subcritical cases (see Lei and Li [7] for detail).

- (b) Theorem 1.1 is still true if we consider (1.10) with  $0 < a \leq c_1(x), c_2(x) \leq b < \infty$  and  $x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) \geq 0$ . And the proof is highly similar to the case  $c_1 = c_2 = 1$ . So in this paper, we only prove for  $c_1 = c_2 = 1$ .

The complete solution of the Lane-Emden conjecture may be a longstanding work. Hence, it will be interesting to consider the Lane-Emden conjecture under some conditions weaker than (1.8).

**Open problem 1.** Can we prove the Lane-Emden conjecture under the following pointwise asymptotic:

$$|v(x)| \leq C|x|^{-\gamma}, \quad \text{for some } 0 < \gamma < \beta.$$

**Open problem 2.** Can we prove the Lane-Emden conjecture under the following integral asymptotic:

$$\int_{B_R} v^s \leq CR^\delta, \quad \text{for some } s > 0, \quad 0 < \delta < 1.$$

Clearly, if problem 2 is solved, problem 1 directly follows by choosing sufficiently large  $s$ .

The paper is organized as follows. In Section 2, we provide a few preliminary results. Some simplified proofs are given for the completeness and convenience of readers. One of the difficulty in the proof of Theorem 1.1 was to control the embedding index, and we derived a varied form of  $W^{2,p}$ -estimate (see Lemma 2.4) to solve this problem. In Section 3, we give the proof of Theorem 1.1. Our proof by classifying the argument into two cases hopefully can deliver the idea and the structure of the proof to readers in a clearer way.

## 2 Preliminaries

Throughout this paper, the standard Sobolev embedding on  $\mathbb{S}^{n-1}$  is frequently used. Here we make some conventions about the notations. Let  $D$  denote the gradient with respect to standard metric on manifold. Let  $n \geq 2, j \geq 1$  be integers and  $1 \leq z_1 < \lambda \leq +\infty, z_2 \neq (n-1)/j$ . For  $u = u(\theta) \in W^{j, z_1}(\mathbb{S}^{n-1})$ , we have

$$\|u\|_{L^{z_2}(\mathbb{S}^{n-1})} \leq C \left( \|D_\theta^j u\|_{L^{z_1}(\mathbb{S}^{n-1})} + \|u\|_{L^1(\mathbb{S}^{n-1})} \right), \quad (2.1)$$

where

$$\begin{cases} \frac{1}{z_2} = \frac{1}{z_1} - \frac{j}{n-1}, & \text{if } z_1 < (n-1)/j, \\ z_2 = \infty, & \text{if } z_1 > (n-1)/j, \end{cases}$$

and  $C = C(j, z_1, n) > 0$ . Although  $C$  may be different from line to line, we always denote the universal constant by  $C$ . For simplicity, in what follows, for a function  $f(r, \theta)$ , we define

$$\|f\|_p(r) = \|f(r, \cdot)\|_{L^p(\mathbb{S}^{n-1})}, \quad (2.2)$$

if no risk of confusion arises. Also let  $s, p, q$  be defined as in Theorem 1.1 and

$$l = s/p, \quad k = \frac{p+1}{p}, \quad m = \frac{q+1}{q}.$$

By Remark 1.2 (b) and Lemma 2.5, throughout the paper, we always assume  $p \geq q$ . At last, we set

$$F(R) = \int_{B_R} u^{q+1} dx.$$

## 2.1 Basic Inequalities

Let us start with a basic yet important fact. Considering  $L^t$ -norm on  $B_{2R}$ , we can write

$$\|f\|_{L^t(B_{2R})}^t = \int_0^{2R} \|f(r)\|_{L^t(\mathbb{S}^{n-1})}^t r^{n-1} dr,$$

then by a standard measurement argument (cf. [17], [19]) one can prove that:

**Lemma 2.1.** *Let  $f_i \in L_{loc}^{p_i}(\mathbb{R}^n)$ , and  $i = 1, \dots, N$ , then for any  $R > 0$ , there exists  $\tilde{R} \in [R, 2R]$  such that*

$$\|f_i\|_{L^{p_i}(\mathbb{S}^{n-1})}(\tilde{R}) \leq (N+1)R^{-\frac{n}{p_i}} \|f_i\|_{L^{p_i}(B_{2R})}, \text{ for each } i = 1, \dots, N.$$

The following lemma is a varied  $W^{2,p}$ -estimate which seems not to appear in any literature, so we give a simple proof.

**Lemma 2.2.** *Let  $1 < \gamma < +\infty$  and  $R > 0$ . For  $u \in W^{2,\gamma}(B_{2R})$ , we have*

$$\|D^2 u\|_{L^\gamma(B_R)} \leq C \left( \|\Delta u\|_{L^\gamma(B_{2R})} + R^{\frac{n}{\gamma} - (n+2)} \|u\|_{L^1(B_{2R})} \right)$$

where  $C = C(\gamma, n) > 0$ .

Proof. First we deal with functions in  $C^2(B_2) \cap C^0(\overline{B_2})$ . By standard elliptic  $W^{2,p}$ -estimate, we have

$$\|D^2 u\|_{L^\gamma(B_1)} \leq C(\|\Delta u\|_{L^\gamma(B_{\frac{3}{2}})} + \|u\|_{L^\gamma(B_{\frac{3}{2}})}). \quad (2.3)$$

By Lemma 2.1,  $\exists \tilde{R} \in [\frac{7}{4}, 2]$  such that on  $B_{\tilde{R}}$ ,  $u$  can be written as  $u = w_1 + w_2$ , where respectively  $w_1$  and  $w_2$  are solutions to

$$\begin{cases} \Delta w_1 = \Delta u, & \text{in } B_{\tilde{R}}, \\ w_1 = 0, & \text{on } \partial B_{\tilde{R}}, \end{cases}$$

and

$$\begin{cases} \Delta w_2 = 0, & \text{in } B_{\tilde{R}}, \\ w_2 = u, & \text{on } \partial B_{\tilde{R}}, \end{cases}$$

and additionally,

$$\int_{\partial B_{\tilde{R}}} u d\sigma \leq C \|u\|_{L^1(B_2)}. \quad (2.4)$$

By standard  $W^{2,p}$ -estimate with homogeneous boundary condition, we have

$$\|w_1\|_{L^\gamma(B_{\frac{3}{2}})} \leq \|w_1\|_{W^{2,\gamma}(B_{\frac{3}{2}})} \leq C \|\Delta w_1\|_{L^\gamma(B_{\tilde{R}})}.$$

Since  $w_2$  can be solved explicitly by Poisson formula on  $B_{\tilde{R}}$ , we see that by (2.4) for any  $x \in B_{\frac{3}{2}} \subsetneq B_{\tilde{R}}$ ,  $w_2(x)$  can be bounded pointwisely by

$$|w_2(x)| \leq C \int_{\partial B_{\tilde{R}}} |u| \leq C \|u\|_{L^1(B_2)}.$$

So,

$$\|w_2\|_{L^\gamma(B_{\frac{3}{2}})} \leq C \|u\|_{L^1(B_2)}.$$

Hence,

$$\begin{aligned} \|u\|_{L^\gamma(B_{\frac{3}{2}})} &\leq \|w_1\|_{L^\gamma(B_{\frac{3}{2}})} + \|w_2\|_{L^\gamma(B_{\frac{3}{2}})} \\ &\leq C(\|\Delta u\|_{L^\gamma(B_{\tilde{R}})} + \|u\|_{L^1(B_2)}). \end{aligned}$$

Therefore, (2.3) becomes

$$\|D^2 u\|_{L^\gamma(B_1)} \leq C(\|\Delta u\|_{L^\gamma(B_2)} + \|u\|_{L^1(B_2)}).$$

Then the lemma follows from a dilation and approximation argument.  $\square$

**Lemma 2.3** (Interpolation inequality on  $B_R$ ). *Let  $1 \leq \gamma < +\infty$  and  $R > 0$ . For  $u \in W^{2,\gamma}(B_R)$ , we have*

$$\|D_x u\|_{L^1(B_R)} \leq C \left( R^{n(1-\frac{1}{\gamma})+1} \|D_x^2 u\|_{L^\gamma(B_R)} + R^{-1} \|u\|_{L^1(B_R)} \right),$$

where  $C = C(\gamma, n) > 0$ .

## 2.2 Pohožaev Identity, Comparison Principle and Energy Estimates

For system (1.1) we have a Rellich-Pohožaev identity, which is the starting point of the proof of Theorem 1.1,

**Lemma 2.4.** *Let  $d_1, d_2 \geq 0$  and  $d_1 + d_2 = n - 2$ , then*

$$\begin{aligned} &\int_{B_R} \left( \frac{n}{p+1} - d_1 \right) v^{p+1} + \left( \frac{n}{q+1} - d_2 \right) u^{q+1} dx \\ &= R^n \int_{\mathbb{S}^{n-1}} \frac{v^{p+1}(R)}{p+1} + \frac{u^{q+1}(R)}{q+1} d\sigma + R^{n-1} \int_{\mathbb{S}^{n-1}} d_1 v' u + d_2 u' v d\sigma + R^n \int_{\mathbb{S}^{n-1}} (v' u' - R^{-2} \nabla_\theta u \cdot \nabla_\theta v) d\sigma. \end{aligned}$$

**Lemma 2.5** (Comparison Principle). *Let  $p \geq q > 0, pq > 1$  and  $(u, v)$  be a positive bounded solution of (1.1). Then we have the following comparison principle,*

$$v^{p+1}(x) \leq \frac{p+1}{q+1} u^{q+1}(x), \quad x \in \mathbb{R}^n.$$

Proof. Let  $l = (\frac{p+1}{q+1})^{\frac{1}{p+1}}$ ,  $\sigma = \frac{q+1}{p+1}$ . So  $l^{p+1}\sigma = 1$ , and  $\sigma \leq 1$ . Denote

$$\omega = v - l u^\sigma.$$

We will show that  $\omega \leq 0$ .

$$\begin{aligned}
\Delta\omega &= \Delta v - l\nabla \cdot (\sigma u^{\sigma-1} \nabla u) \\
&= \Delta v - l\sigma(\sigma-1)|\nabla u|^2 - l\sigma u^{\sigma-1} \Delta u \\
&\geq -u^q + l\sigma u^{\sigma-1} v^p \\
&= u^{\sigma-1} \left( \left( \frac{v}{l} \right)^p - u^{q+1-\sigma} \right) \\
&= u^{\sigma-1} \left( \left( \frac{v}{l} \right)^p - u^{\sigma p} \right).
\end{aligned}$$

So,  $\Delta\omega > 0$  if  $w > 0$ . Now, suppose  $w > 0$  for some  $x \in \mathbb{R}^n$ , and there are two cases:

Case 1:  $\exists x_0 \in \mathbb{R}^n$ , such that  $\omega(x_0) = \max_{\mathbb{R}^n} \omega(x) > 0$ , and  $\Delta\omega(x_0) \leq 0$ . However, when  $w > 0$ ,  $\Delta\omega > 0$ , a contradiction.

Case 2: There exists a sequence  $\{x_m\}$  with  $|x_m| \rightarrow +\infty$ , such that  $\lim_{m \rightarrow +\infty} \omega(x_m) = \max_{\mathbb{R}^n} \omega(x) > c_0 > 0$  for some constant  $c_0$ .

Let  $\omega_R(x) = \phi(\frac{x}{R})\omega(x)$ , where  $\phi(x) \in C_0^\infty(B_1)$  is a cutoff function and  $\phi(x) \equiv 1$  in  $B_{\frac{1}{2}}$ . Since  $\omega_R(x) = 0$  on  $\partial B_R$ , there exists an  $x_R \in B_R$  such that  $\omega_R(x_R) = \max_{B_R} \omega_R(x)$  and  $\lim_{R \rightarrow +\infty} \omega(x_R) = \max_{\mathbb{R}^n} \omega(x) > 0$ . Also,

$$0 = \nabla\omega_R(x_R) = \phi\left(\frac{x_R}{R}\right)\nabla\omega(x_R) + \frac{1}{R}\nabla\phi\left(\frac{x_R}{R}\right)\omega(x_R).$$

As  $\phi(\frac{x_R}{R}) \geq c_1 > 0$  for some constant  $c_1$  (in fact,  $\phi(\frac{x_R}{R}) \rightarrow 1$ ) and  $\omega(x_R)$  is bounded since  $u, v$  are bounded in  $\mathbb{R}^n$ , we see that  $\nabla\omega(x_R) \rightarrow 0$  as  $R \rightarrow +\infty$ . So,

$$\begin{aligned}
0 &\geq \Delta\omega_R(x_R) = \frac{1}{R^2}\Delta\phi\left(\frac{x_R}{R}\right)\omega(x_R) + \frac{2}{R}\nabla\phi\left(\frac{x_R}{R}\right) \cdot \nabla\omega(x_R) + \phi\left(\frac{x_R}{R}\right)\Delta\omega(x_R) \\
&\Rightarrow 0 \geq \Delta\omega(x_R) + O\left(\frac{1}{R^2}\right)
\end{aligned}$$

Since  $\omega(x_R) > c_0/2$  for sufficiently large  $R$ ,  $\Delta\omega(x_R) > c_2 > 0$  for some constant  $c_2$ , a contradiction.  $\square$

**Remark 2.6.** For general Lane-Emden type system (1.10), we can choose

$$w = v - Clu^\sigma, \quad \text{where} \quad C^{p+1} = \sup_{x \in \mathbb{R}^n} \frac{c_2(x)}{c_1(x)}.$$

By the same arguments, we can also get the desired comparison principle.

Next we prove a group of energy estimates which are crucial to the entire argument in this paper. As Theorem 1.1 points out, better energy estimates are the key to the Lane-Emden conjecture. Unfortunately, efforts have been made so far only provide the following inequalities, which are first obtained by Serrin and Zou [17] (1996). Here we give a simpler proof than the original one for the convenience of readers.

**Lemma 2.7.** *Let  $p, q > 0$  with  $pq > 1$ . For any positive solution  $(u, v)$  of (1.1)*

$$\int_{B_R} u \leq CR^{n-\alpha}, \text{ and } \int_{B_R} v \leq CR^{n-\beta}, \quad (2.5)$$

$$\int_{B_R} u^q \leq CR^{n-q\alpha}, \text{ and } \int_{B_R} v^p \leq CR^{n-p\beta}. \quad (2.6)$$

Proof. Without loss of generality, we can assume that  $p \geq q$ .

Let  $\phi \in C^\infty(B_R(0))$  be the first eigenfunction of  $-\Delta$  in  $B_R$  and  $\lambda$  be the eigenvalue. By definition and rescaling, it is easy to see that  $\phi|_{\partial B_R} = 0$  and  $\lambda \sim \frac{1}{R^2}$ . By normalizing, one gets  $\phi \geq c_0 > 0$  on  $B_{R/2}$  for some constant  $c_0$  independent of  $R$ ,  $\phi(0) = \|\phi\|_\infty = 1$ . So, multiplying (1.1) by  $\phi$  then integrating by parts on  $B_R$  we have,

$$\int_{B_R} \phi u^q = - \int_{B_R} \phi \Delta v = \int_{\partial B_R} v \frac{\partial \phi}{\partial n} d\sigma + \lambda \int_{B_R} \phi v.$$

By Hopf's Lemma we know that  $\frac{\partial \phi}{\partial n} < 0$  on  $\partial B_R$ , so

$$\int_{B_R} \phi u^q \leq \lambda \int_{B_R} \phi v. \quad (2.7)$$

Similarly, we have

$$\int_{B_R} \phi v^p \leq \lambda \int_{B_R} \phi u. \quad (2.8)$$

Applying Lemma 2.5 to (2.7), we have

$$\frac{1}{R^2} \int_{B_R} \phi v \geq C \int_{B_R} \phi v^{\frac{q(p+1)}{q+1}}.$$

Notice that  $\frac{q(p+1)}{q+1} > 1$  as  $pq > 1$ , so by Hölder inequality

$$\begin{aligned} \int_{B_R} \phi v^{\frac{q(p+1)}{q+1}} &\geq \left( \int_{B_R} \phi v \right)^{\frac{q(p+1)}{q+1}} \left( \int_{B_R} \phi \right)^{-\left(\frac{q(p+1)}{q+1}-1\right)} \\ &\geq C \left( \int_{B_R} \phi v \right)^{\frac{q(p+1)}{q+1}} R^{-n \frac{qp-1}{q+1}}. \end{aligned}$$

So,

$$\begin{aligned} \frac{1}{R^2} \int_{B_R} \phi v &\geq C \left( \int_{B_R} \phi v \right)^{\frac{q(p+1)}{q+1}} R^{-n \frac{qp-1}{q+1}} \\ \Rightarrow \int_{B_R} \phi v &\leq C R^{n-\beta}. \end{aligned}$$

Therefore, by (2.7)

$$\int_{B_R} \phi u^q \leq C R^{n-\beta-2} = C R^{n-q\alpha}.$$

Now, **Case 1:** If  $q > 1$ , then by Hölder inequality

$$\int_{B_R} \phi u \leq \left( \int_{B_R} \phi u^q \right)^{\frac{1}{q}} \left( \int_{B_R} \phi \right)^{\frac{1}{q'}} \leq C R^{\frac{n}{q}-\alpha} R^{\frac{n}{q'}} = C R^{n-\alpha}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Mean while, by (2.8)

$$\int_{B_R} \phi v^p \leq C R^{n-\alpha-2} = C R^{n-p\beta}.$$

This finishes the proof for Case 1.

**Case 2:** Assume that  $q \leq 1$ . To prove this trickier case, we begin with a lemma of energy-type estimate,



**Lemma 2.8.** *If  $\Delta u \leq 0$ , then for  $\gamma \in (0, 1)$ ,  $\eta \in C_0^\infty(\mathbb{R}^n)$ ,*

$$\int_{\mathbb{R}^n} \frac{4}{\gamma^2} |D(u^{\frac{\gamma}{2}})|^2 \eta^2 = \int \eta^2 |Du|^2 u^{\gamma-2} \leq C \int |D\eta|^2 u^\gamma. \quad (2.9)$$

Proof. Multiply  $\eta^2 u^{\gamma-1}$  to  $\Delta u \leq 0$  then integrate over the whole space.  $\square$

We rewrite (2.9) as

$$\int_{B_R} |Du|^2 u^{\gamma-2} \leq \frac{C_\gamma}{R^2} \int_{B_{2R}} u^\gamma \quad (2.10)$$

where  $C_\gamma \rightarrow +\infty$  as  $\gamma \rightarrow 1$ . From Poincaré's Inequality, we have

$$|f|_{\frac{na}{n-a}, \Omega_R} \leq C(n, a, \Omega) \left( |Df|_{a, \Omega_R} + |R|^{\frac{n-a}{a}} |f_{\Omega_R}| \right), \quad (2.11)$$

where

$$f_{\Omega_R} = \oint_{\Omega_R} f = \frac{1}{|\Omega_R|} \int_{\Omega_R} f, \quad \Omega_R = \{Rx | x \in \Omega\}.$$

Next we prove a variation of embedding inequality,

**Lemma 2.9.** *For any  $l \geq 1$ ,*

$$|f^l|_{\frac{an}{n-a}, \Omega_R} \leq C(n, a, \Omega) \left( |D(f^l)|_{a, \Omega_R} + |R|^{\frac{n-a}{a}} |f_{\Omega_R}|^l \right) \quad (2.12)$$

Proof. By (2.11),

$$\begin{aligned} |f^l|_{\frac{an}{n-a}, \Omega_R} &\leq C(n, a, \Omega) \left( |D(f^l)|_{a, \Omega_R} + |R|^{\frac{n-a}{a}} |(f^l)_{\Omega_R}| \right) \\ &\leq C(n, a, \Omega) \left( |D(f^l)|_{a, \Omega_R} + |R|^{\frac{n-a}{a}-n} \int_{\Omega_R} f^l dx \right) \\ &\leq C(n, a, \Omega) \left\{ |D(f^l)|_{a, \Omega_R} + |R|^{\frac{n-a}{a}-n} \left( \int_{\Omega_R} f dx \right)^{\theta l} \left( \int_{\Omega_R} f^{\frac{an}{n-a}} dx \right)^{(1-\theta)l \frac{n-a}{lan}} \right\}, \quad \theta = \frac{1 - \frac{n-a}{na}}{l - \frac{n-a}{na}} \\ &\leq C(n, a, \Omega) |D(f^l)|_{a, \Omega_R} + \frac{1}{2} |f^l|_{\frac{an}{n-a}, \Omega_R} + C(n, a, \Omega) |R|^{\frac{n-a}{a}} |f_{\Omega_R}|^l. \end{aligned}$$

In getting the last inequality, we have used the Young's inequality. So we get (2.12).  $\square$

Let  $l \geq 1$ ,  $\theta \leq 2q < 2$ ,  $\gamma = l\theta < 1$ ,  $f = u^{\frac{\theta}{2}}$ ,  $a = 2$ . Then

$$\begin{aligned} |f^l|_{\frac{2n}{n-2}, B_R} &\leq C \left( |Df^l|_{2, B_R} + R^{\frac{n-2}{2}} |f_{B_R}|^l \right) \\ &\leq C \left( |D(u^{\frac{l\theta}{2}})|_{2, B_R} + R^{\frac{n-2}{2}} |(u^{\frac{\theta}{2}})_{B_R}|^l \right) \\ &\leq \frac{C}{R} \left( \int_{B_{2R}} u^{l\theta} \right)^{\frac{1}{2}} + R^{\frac{n-2}{2}} \left( \int_{B_R} u^{\frac{\theta}{2}} \right)^l. \end{aligned} \quad (2.13)$$

The last term on the right can be estimate by Hölder and the fact that  $\int_{B_R} u^q \leq CR^{-q\alpha}$  since  $\frac{\theta}{2} < q$ . This yields that

$$\int_{B_R} u^{\frac{n}{n-2}\theta l} \leq C \left( R^{-\frac{2n}{n-2}} \left( \int_{B_{2R}} u^{l\theta} \right)^{\frac{n}{n-2}} + R^{n - \frac{n}{n-2}l\theta\alpha} \right). \quad (2.14)$$

This means if  $\int_{B_R} u^{l\theta} \leq CR^{-l\theta\alpha}$ , we have  $\int_{B_R} u^{\frac{n}{n-2}l\theta} \leq CR^{-\frac{n}{n-2}l\theta\alpha}$  provided  $l\theta < 1$ . By  $\int_{B_R} u^q \leq CR^{-q\alpha}$ , one gets

$$\int_{B_R} u^s \leq C(s)R^{-s\alpha}, \quad \text{for } s < \frac{n}{n-2} \quad (2.15)$$

where  $C(s) \rightarrow +\infty$  as  $s \rightarrow \frac{n}{n-2}$ .

By taking  $s = 1$ , the above inequality immediately leads to

$$\int_{B_R} u \leq CR^{n-\alpha}.$$

Since  $pq > 1$  and we assume that  $p \geq q$ ,  $q$  must be greater than 1, then by Hölder and (2.8) we get

$$\int_{B_R} v^p \leq CR^{n-p\beta}.$$

This finishes the proof of Lemma 2.7.  $\square$

### 2.3 Key Estimates on $\mathbb{S}^{n-1}$

Now that we have energy inequalities (2.6), in our assumption (1.8) we can always assume  $s \geq p$ . Since  $l = \frac{s}{p}$ , we have  $l \geq 1$ . The following estimates for quantities on sphere  $\mathbb{S}^{n-1}$  are necessary to the proof.

**Proposition 2.10.** *For  $R \geq 1$ , there exists  $\tilde{R} \in [R, 2R]$  such that for  $l = \frac{s}{p} \geq 1$ ,  $k = \frac{p+1}{p}$  and  $m = \frac{q+1}{q}$ , we have*

$$\begin{aligned} \|u\|_1(\tilde{R}) &\leq CR^{-\alpha}, \quad \|v\|_1(\tilde{R}) \leq CR^{-\beta}, \\ \|D_x^2 u\|_l(\tilde{R}) &\leq CR^{-\frac{lp\beta}{l+\varepsilon}}, \quad \|D_x^2 v\|_{1+\varepsilon}(\tilde{R}) \leq CR^{-\frac{q\alpha}{1+\varepsilon}}, \\ \|D_x u\|_1(\tilde{R}) &\leq CR^{1-\frac{\alpha+2}{1+\varepsilon}}, \quad \|D_x v\|_1(\tilde{R}) \leq CR^{1-\frac{\beta+2}{1+\varepsilon}}, \\ \|D_x^2 u\|_k(\tilde{R}) &\leq C(R^{-n}F(2R))^{\frac{1}{k}}, \quad \|D_x^2 v\|_m(\tilde{R}) \leq C(R^{-n}F(2R))^{\frac{1}{m}}. \end{aligned}$$

In view of Lemma 2.1, to prove Proposition 2.10, we shall give the corresponding estimates on  $B_{2R}$  first. We use the varied  $W^{2,p}$ -estimate (i.e. Lemma 2.2) to achieve this.

**Proposition 2.11.** *For  $R \geq 1$ , we have*

$$\begin{cases} \|u\|_{L^1(B_R)} &\leq CR^{n-\beta}, \\ \|v\|_{L^1(B_R)} &\leq CR^{n-\alpha}, \end{cases} \quad (2.16)$$

$$\begin{cases} \|D_x^2 u\|_{L^{l+\varepsilon}(B_R)}^{l+\varepsilon} &\leq CR^{n-lp\beta}, \quad \text{with } l = \frac{s}{p} \geq 1, \\ \|D_x^2 v\|_{L^{1+\varepsilon}(B_R)}^{1+\varepsilon} &\leq CR^{n-q\alpha}, \end{cases} \quad (2.17)$$

$$\begin{cases} \|D_x u\|_{L^1(B_R)} &\leq CR^{n+1-\frac{\alpha+2}{1+\varepsilon}}, \\ \|D_x v\|_{L^1(B_R)} &\leq CR^{n+1-\frac{\beta+2}{1+\varepsilon}}, \end{cases} \quad (2.18)$$

and let  $k = \frac{p+1}{p}$ ,  $m = \frac{q+1}{q}$ ,

$$\begin{cases} \|D_x^2 u\|_{L^k(B_R)}^k &\leq CF(2R), \\ \|D_x^2 v\|_{L^m(B_R)}^m &\leq CF(2R). \end{cases} \quad (2.19)$$

Proof. Some frequently used facts include,  $q\alpha = \beta + 2$ ,  $p\beta = \alpha + 2$  and hence  $n - kp\beta < 0$  (due to (1.4)) and therefore  $l < k$  (since  $n - lp\beta \geq 0$ ).

Estimates (2.16) directly follow from (2.5) in Lemma 2.7.

For the first estimate of (2.17), after applying Lemma 2.2, the mixed type  $W^{2,p}$ -estimate<sup>1</sup>, we get

$$\|D_x^2 u\|_{L^{l+\varepsilon}(B_R)}^{l+\varepsilon} \leq C \left( \|\Delta u\|_{L^{l+\varepsilon}(B_{2R})}^{l+\varepsilon} + R^{n-(l+\varepsilon)(n+2)} \|u\|_{L^1(B_{2R})}^{l+\varepsilon} \right).$$

Then we use the assumed estimate (1.8) and Lemma 2.7 to get

$$\begin{aligned} \|D_x^2 u\|_{L^{l+\varepsilon}(B_R)}^{l+\varepsilon} &\leq C \left( \int_{B_{2R}} v^{p(l+\varepsilon)} dx + R^{n-(l+\varepsilon)(n+2)} R^{(l+\varepsilon)(n-\alpha)} \right) \\ &\leq C \left( R^{n-pl\beta} + R^{n-(l+\varepsilon)(2+\alpha)} \right) \\ &\leq CR^{n-pl\beta}, \end{aligned}$$

where the last inequality is due to  $\alpha + 2 = p\beta$ . For the second estimate of (2.17),

$$\begin{aligned} \|D_x^2 v\|_{L^{1+\varepsilon}(B_R)}^{1+\varepsilon} &\leq C \left( \|\Delta v\|_{L^{1+\varepsilon}(B_{2R})}^{1+\varepsilon} + R^{n-(1+\varepsilon)(n+2)} \|v\|_{L^1(B_{2R})}^{1+\varepsilon} \right) \\ &\leq C \left( \int_{B_{2R}} u^{q(1+\varepsilon)} dx + R^{n-(1+\varepsilon)(n+2)} R^{(1+\varepsilon)(n-\beta)} \right) \\ &\leq C \left( R^{n-q\alpha} + R^{n-(1+\varepsilon)(\beta+2)} \right) \\ &\leq CR^{n-q\alpha}. \end{aligned}$$

For the first estimate of (2.18), by Lemma 2.3,

$$\begin{aligned} \|D_x u\|_{L^1(B_R)} &\leq C \left( R^{n(1-\frac{1}{1+\varepsilon})+1} \|D_x^2 u\|_{L^{1+\varepsilon}(B_R)} + R^{-1} \|u\|_{L^1(B_R)} \right) \\ &\leq C \left( R^{n(1-\frac{1}{1+\varepsilon})+1} R^{\frac{n-p\beta}{1+\varepsilon}} + R^{-1} R^{n-\alpha} \right) \\ &\leq CR^{n+1-\frac{\alpha+2}{1+\varepsilon}}, \end{aligned}$$

The second estimate in (2.18) can be obtained by a similar process. Last, the fact that  $n - (p+1)\beta < 0$  gives

$$\begin{aligned} \|D_x^2 u\|_{L^k(B_R)}^k &\leq C \left( \int_{B_{2R}} |\Delta u|^k dx + R^{n-k(n+2)} \left( \int_{B_{2R}} |u| dx \right)^k \right) \\ &\leq C \left( \int_{B_{2R}} v^{p+1} dx + R^{n-k(n+2)} R^{k(n-\alpha)} \right) \\ &\leq C \left( F(2R) + R^{n-(p+1)\beta} \right) \\ &\leq CF(2R), \end{aligned}$$

and hence the first estimate in (2.19) follows, and similarly we get the second estimate.  $\square$

**Proof of Proposition 2.10:** By Lemma 2.1,  $\exists \tilde{R} \in [R, 2R]$ , Proposition 2.10 follows from Proposition 2.11 immediately.  $\square$

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<sup>1</sup>Notice that with the standard  $W^{2,p}$ -estimate, we end up with a term of  $\|u\|_{l+\varepsilon}$  which cannot be estimated by any energy bound.

**Lemma 2.12.** *There exists  $M > 0$  such that  $\exists \{R_j\} \rightarrow +\infty$ ,*

$$F(4R_j) \leq MF(R_j).$$

Proof. Suppose not, then for any  $M > 0$  and any  $\{R_j\} \rightarrow +\infty$ , we have

$$F(4R_j) > MF(R_j).$$

Take  $M > 5^n$  and  $R_{j+1} = 4R_j$  with  $R_0 > 1$ . Therefore,

$$F(R_j) > M^j F(R_0),$$

which leads to a contradiction with  $F(R_j) \leq CR_j^n \leq C(4^j R_0)^n$ .  $\square$

### 3 Proof of Liouville Theorem

From now on, without loss of generality, we may assume  $p \geq q$ . By Lemma 2.5,  $\|v\|_{L^{p+1}(B_R)}^{p+1} \leq \|u\|_{L^{q+1}(B_R)}^{q+1}$ . By the Rellich-Pohožaev type identity in Lemma 2.4, we can denote

$$F(R) := \int_{B_R} u^{q+1} \leq CG_1(R) + CG_2(R), \quad (3.1)$$

where

$$G_1(R) = R^n \int_{\mathbb{S}^{n-1}} u^{q+1}(R) d\sigma, \quad (3.2)$$

$$G_2(R) = R^n \int_{\mathbb{S}^{n-1}} (|D_x u(R)| + R^{-1}u(R))(|D_x v(R)| + R^{-1}v(R)) d\sigma. \quad (3.3)$$

Heuristically, we are aiming for estimate as

$$G_i(R) \leq CR^{-a_i} F^{1-\delta_i}(4R), \quad i = 1, 2. \quad (3.4)$$

Then by Lemma 2.12 there exists a sequence  $\{R_j\} \rightarrow +\infty$  such that

$$G_i(R_j) \leq CR^{-a_i} F^{1-\delta_i}(R_j), \quad i = 1, 2.$$

Suppose there are infinitely many  $R_j$ 's such that  $G_1(R_j) \geq G_2(R_j)$ , then take that subsequence of  $\{R_j\}$  and still denote as  $\{R_j\}$ . We do the same if there are infinitely many  $R_j$ 's such that  $G_1(R_j) \leq G_2(R_j)$ . So, there are only two cases we shall deal with: there exists a sequence  $\{R_j\} \rightarrow +\infty$  such that

Case 1:  $G_1(R_j) \geq G_2(R_j)$ . Then we prove  $a_1 > 0$ ,  $\delta_1 > 0$ . So,  $F^{\delta_1}(R_j) \leq CR_j^{-a_1} \rightarrow 0$ ,

Case 2:  $G_1(R_j) \leq G_2(R_j)$ . Then we prove  $a_2 > 0$ ,  $\delta_2 > 0$ . So,  $F^{\delta_2}(R_j) \leq CR_j^{-a_2} \rightarrow 0$ .

Then we conclude that  $F(R) \equiv 0$ .

Surprisingly, for both cases  $a_i \approx (\alpha + \beta + 2 - n)\delta_i$ . Indeed, we have

**Theorem 3.1.** *For  $F(R)$  defined as (3.1) and  $\alpha, \beta$  defined as (1.3), there exists a sequence  $\{R_j\} \rightarrow +\infty$  such that*

$$F(R_j) \leq CR_j^{-(\alpha+\beta+2-n)+o(1)}.$$

Hence, Theorem 1.1 is a direct consequence of Theorem 3.1, and we only need to prove Theorem 3.1 for case 1 and 2.

### 3.1 Case 1: Estimate for $G_1(R)$

According to previous discussion in the introduction, we assume that

$$p \geq q > 0, \quad pq > 1, \quad \beta \leq \alpha < n - 2, \quad n \geq 3,$$

hence in particular

$$p > \frac{n}{n-2}. \quad (3.5)$$

**Remark 3.2.** For systems (1.10) with double bounded coefficients, (3.5) is a necessary condition for existence of positive solution, see [7].

In addition to our assumption that  $n - s\beta < 1$ , since we have energy inequalities (2.6), we can assume  $s \geq p$ . Also, if  $n - s\beta < 0$ , (1.8) implies  $v \equiv 0$  and hence  $u \equiv 0$ . So, we assume  $n - s\beta \geq 0$ . Let  $l = \frac{s}{p}$ , then

$$l \geq 1, \text{ and } \frac{n-1}{p\beta} < l \leq \frac{n}{p\beta}. \quad (3.6)$$

It is worthy to point out that, what the proof of Lane-Emden conjecture really needs is a “breakthrough” on the energy estimate (2.6).  $s$  in (1.8) needs not be very large but enough to satisfy  $n - s\beta < 1$ . In other words,  $s$  can be very close to  $\frac{n-1}{\beta}$ , and it is sufficient to prove Theorem 1.1.

The strategies of attacking  $G_1$  and  $G_2$  are the same. Basically, first by Hölder inequality we split the quantities on sphere  $\mathbb{S}^{n-1}$  into two parts. One has a lower (than original) index after embedding, and the other has a higher one. Then we estimate the latter part by  $F(R)$ , and thus we get a feedback estimate as (3.4).

Let

$$k = \frac{p+1}{p}.$$

Since  $p\beta = \alpha + 2$ ,  $n - (p+1)\beta = n - 2 - (\alpha + \beta) < 0$  by (1.4). Thus,  $n - kp\beta < 0$  as  $n - lp\beta \geq 0$ , it follows that  $l < k$ .

**Subcase 1.1**  $\frac{1}{l} \geq \frac{2}{n-1} + \frac{1}{q+1}$ .

Note that in this subcase, since  $l \geq 1$ , we must have  $n \geq 4$  (i.e.,  $n \neq 3$ ). By (3.5) we see that  $k = 1 + \frac{1}{p} < 1 + \frac{n-2}{n} = \frac{2}{n}(n-1) \leq \frac{n-1}{2}$ . Take

$$\frac{1}{\mu} = \frac{1}{k} - \frac{2}{n-1}.$$

So,  $W^{2,k}(\mathbb{S}^{n-1}) \hookrightarrow L^\mu(\mathbb{S}^{n-1})$ .

Take

$$\frac{1}{\lambda} = \frac{1}{l} - \frac{2}{n-1} \geq \frac{1}{q+1}.$$

Then  $W^{2,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^\lambda(\mathbb{S}^{n-1})$ .

Direct verification shows that  $\frac{1}{\mu} = \frac{1}{k} - \frac{2}{n-1} \leq \frac{1}{q+1}$  which is due to (1.2), so we have

$$\frac{1}{\mu} \leq \frac{1}{q+1} \leq \frac{1}{\lambda}.$$

Then by Hölder inequality and Sobolev embedding (2.1), we have (with notation (2.2))

$$\|u\|_{q+1}(R) \leq \|u\|_{\lambda}^{\theta} \|u\|_{\mu}^{1-\theta}(R) \quad (3.7)$$

$$\leq C(R^2 \|D_x^2 u\|_{l+\varepsilon}(R) + \|u\|_1(R))^{\theta} (R^2 \|D_x^2 u\|_k(R) + \|u\|_1(R))^{1-\theta}, \quad (3.8)$$

where  $\theta \in [0, 1]$  and

$$\frac{1}{q+1} = \frac{\theta}{\lambda} + \frac{1-\theta}{\mu}. \quad (3.9)$$

Since  $l$  can be 1 (then  $W^{2,p}$ -estimate fails for  $\|D_x^2 u\|_1(R)$ ), we add an  $\varepsilon$  to  $l$  for later use of  $W^{2,p}$ -estimate.  $\varepsilon$  can be any real positive number and later will be chosen sufficiently small.

To get desired estimate, we have requirements in form of inequalities involving parameters, such as  $\alpha, \beta, \varepsilon$  and etc. To verify those requirements very often we just verify strict inequalities with  $\varepsilon = 0$  because such inequalities continuously depend on  $\varepsilon$ .

So, by (3.2) and (3.8)

$$G_1(R) \leq CR^n \left( (R^2 \|D_x^2 u\|_{l+\varepsilon}(R) + \|u\|_1(R))^{\theta} (R^2 \|D_x^2 u\|_k(R) + \|u\|_1(R))^{1-\theta} \right)^{q+1}. \quad (3.10)$$

Then by Proposition 2.10, there exists  $\tilde{R} \in [R, 2R]$  such that

$$\begin{aligned} G_1(\tilde{R}) &\leq CR^n \left( (R^2 R^{-\frac{lp\beta}{l+\varepsilon}} + R^{-2-\alpha})^{\theta} (R^2 (R^{-n} F(4R))^{\frac{1}{k}} + R^{-\alpha})^{1-\theta} \right)^{q+1} \\ &\leq CR^n \left( R^{2-\frac{lp\beta\theta}{l+\varepsilon}-\frac{n(1-\theta)}{k}} F^{\frac{1-\theta}{k}}(4R) \right)^{q+1} \\ &\leq R^{-a_1} F^{1-\delta_1}(4R), \end{aligned}$$

where the last inequality is due to  $R^{-\frac{n}{k}} > R^{-\alpha-2}$  and

$$a_1 = a_1^{\varepsilon} = (q+1) \left( \frac{lp\beta\theta}{l+\varepsilon} + \frac{np(1-\theta)}{p+1} - 2 - \frac{n}{1+q} \right), \quad (3.11)$$

$$1 - \delta_1 = \frac{(1-\theta)p(q+1)}{p+1}. \quad (3.12)$$

Since for sufficiently small  $\varepsilon$ ,  $a_1^{\varepsilon} > 0$  and  $\delta_1 > 0$  are just a perturbation of

$$a_1^0 > 0, \text{ and } \delta_1 > 0, \quad (3.13)$$

we only need to prove (3.13) is true.

Since  $lp = s$ ,  $p\beta = \alpha + 2$  and  $q\alpha = \beta + 2$ ,

$$\begin{aligned} a_1^0 &= p\beta\theta(q+1) + (1-\delta_1)n - 2(q+1) - n \\ &= (q+1)(p\beta\theta - 2) - \delta_1 n \\ &= (q+1)(p\beta(\theta - 1) + p\beta - 2) - \delta_1 n \\ &= (q+1)(-\alpha(1-\delta_1) + \alpha) - \delta_1 n \\ &= \delta_1((q+1)\alpha - n) \\ &= (\alpha + \beta + 2 - n)\delta_1. \end{aligned}$$

So we just need to prove  $\delta_1 > 0$ . By (3.11) and (3.9) we have

$$\begin{aligned}
& (1 - \theta)p(q + 1) < p + 1 \\
& \Leftrightarrow \frac{\frac{1}{l} - \frac{2}{n-1} - \frac{1}{1+q}}{\frac{1}{l} - \frac{1}{k}}(q + 1) < k \\
& \Leftrightarrow \left(\frac{1}{l} - \frac{2}{n-1}\right)(q + 1) - 1 < \frac{k}{l} - 1 \\
& \Leftrightarrow \frac{1}{l}(q + 1 - 1 - \frac{1}{p}) < \frac{2}{n-1}(q + 1) \\
& \Leftrightarrow \frac{pq - 1}{s} < \frac{2(q + 1)}{n - 1} \\
& \Leftrightarrow n - 1 < s\beta,
\end{aligned}$$

and the last inequality is included in our assumption. So, we have proved subcase 1.1.

**Subcase 1.2**  $\frac{1}{l} < \frac{2}{n-1} + \frac{1}{q+1}$ .

As discussed in the beginning of subcase 1.1,  $k < \frac{n-1}{2}$  if  $n > 3$ . Since  $l < k$ ,  $\frac{1}{l} > \frac{2}{n-1}$  for  $n > 3$ . When  $n = 3$ , since  $l \geq 1$  by (3.6),  $\frac{1}{l} \leq 1 = \frac{2}{n-1}$ .

Therefore, for  $n > 3$ , take

$$\frac{1}{\lambda} = \frac{1}{l} - \frac{2}{n-1} < \frac{1}{q+1},$$

and for  $n = 3$ , take

$$\lambda = \infty,$$

so we have

$$W^{2,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^\lambda(\mathbb{S}^{n-1}), \quad n \geq 3.$$

So,

$$\|u\|_{q+1}(R) \leq C\|u\|_\lambda(R) \leq C(R^2\|D_x^2 u\|_{l+\varepsilon}(R) + \|u\|_1(R)).$$

Therefore, by Proposition 2.10 there exists  $\tilde{R} \in [R, 2R]$  such that

$$G_1(\tilde{R}) \leq CR^n(R^2\|D_x^2 u\|_{l+\varepsilon}(R) + \|u\|_1(R))^{q+1} \quad (3.14)$$

$$\leq CR^n(R^2 R^{-\frac{lp\beta}{l+\varepsilon}} + R^{-\alpha})^{q+1} \quad (3.15)$$

$$\leq CR^{n+(2-\frac{lp\beta}{l+\varepsilon})(q+1)}. \quad (3.16)$$

So,

$$\begin{aligned}
F(\tilde{R}) & \leq CR^{n+(2-\frac{lp\beta}{l+\varepsilon})(q+1)} \\
& \leq CR^{n+(2-p\beta)(q+1)+\frac{\varepsilon p\beta}{l+\varepsilon}(q+1)} \\
& \leq CR^{-(\alpha+\beta+2-n)+\frac{\varepsilon p\beta}{l+\varepsilon}(q+1)}.
\end{aligned}$$

Since  $\varepsilon$  can be arbitrarily small,

$$F(\tilde{R}) \leq CR^{-(\alpha+\beta+2-n)+o(1)}.$$

Thus, we have proved Case 1.

### 3.2 Case 2: Estimate for $G_2(R)$ .

Let

$$m = \frac{q+1}{q}.$$

**Subcase 2.1**  $m < n-1$ .

With  $z, z' > 0$  and  $\frac{1}{z} + \frac{1}{z'} = 1$ , (3.3) becomes,

$$\begin{aligned} G_2(R) &\leq CR^n \| |D_x u| + R^{-1}u \|_z \| |D_x v| + R^{-1}v \|_{z'}(R) \\ &\leq CR^n (\|D_x u\|_z(R) + R^{-1}\|u\|_z(R)) (\|D_x v\|_{z'}(R) + R^{-1}\|v\|_z(R)) \\ &\leq CR^n (\|D_x u\|_z(R) + R^{-1}\|u\|_1(R)) (\|D_x v\|_{z'}(R) + R^{-1}\|v\|_1(R)), \end{aligned} \quad (3.17)$$

where the last inequality is due to

$$\|u\|_z(R) \leq C(R\|D_x u\|_z(R) + \|u\|_1(R)), \text{ and } \|v\|_{z'}(R) \leq C(R\|D_x v\|_{z'}(R) + \|v\|_1(R)).$$

Assume there exists  $z$  (we shall check the existence later) such that by Sobolev Embedding (2.1),

$$\begin{aligned} \|D_x u\|_z(R) &\leq \|D_x u\|_{\rho_1}^{\tau_1}(R) \|D_x u\|_{\gamma_1}^{1-\tau_1}(R) \\ &\leq C(R\|D_x^2 u\|_{l+\varepsilon}(R) + \|D_x u\|_1(R))^{\tau_1} (R\|D_x^2 u\|_k(R) + \|D_x u\|_1(R))^{1-\tau_1}, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \|D_x v\|_{z'}(R) &\leq \|D_x v\|_{\rho_2}^{\tau_2}(R) \|D_x v\|_{\gamma_2}^{1-\tau_2}(R) \\ &\leq C(R\|D_x^2 v\|_{1+\varepsilon}(R) + \|D_x v\|_1(R))^{\tau_2} (R\|D_x^2 v\|_m(R) + \|D_x v\|_1(R))^{1-\tau_2}, \end{aligned} \quad (3.19)$$

where  $\tau_1, \tau_2 \in [0, 1]$  and

$$\frac{1}{z} = \frac{\tau_1}{\rho_1} + \frac{1-\tau_1}{\gamma_1}, \quad (3.20)$$

$$\frac{1}{z'} = \frac{\tau_2}{\rho_2} + \frac{1-\tau_2}{\gamma_2}, \quad (3.21)$$

and since  $l < k \leq m < n-1$ , define

$$\frac{1}{\rho_1} = \frac{1}{l} - \frac{1}{n-1}, \quad \frac{1}{\gamma_1} = \frac{1}{k} - \frac{1}{n-1}, \quad (3.22)$$

$$\frac{1}{\rho_2} = 1 - \frac{1}{n-1}, \quad \frac{1}{\gamma_2} = \frac{1}{m} - \frac{1}{n-1}. \quad (3.23)$$

So, we have

$$\begin{aligned} W^{1,l+\varepsilon}(\mathbb{S}^{n-1}) &\hookrightarrow L^{\rho_1}(\mathbb{S}^{n-1}), \quad W^{1,k}(\mathbb{S}^{n-1}) \hookrightarrow L^{\gamma_1}(\mathbb{S}^{n-1}), \\ W^{1,1+\varepsilon}(\mathbb{S}^{n-1}) &\hookrightarrow L^{\rho_2}(\mathbb{S}^{n-1}), \quad W^{1,m}(\mathbb{S}^{n-1}) \hookrightarrow L^{\gamma_2}(\mathbb{S}^{n-1}). \end{aligned}$$

To verify the existence of such  $z$ , by (3.20)-(3.23), we expect that

$$\max \left\{ \frac{1}{k} - \frac{1}{n-1}, \frac{1}{n-1} \right\} \leq \frac{1}{z} \leq \min \left\{ \frac{1}{l} - \frac{1}{n-1}, \frac{1}{q+1} + \frac{1}{n-1} \right\}. \quad (3.24)$$

Thus, we need to verify, (i)  $\frac{1}{k} - \frac{1}{n-1} \leq \frac{1}{l} - \frac{1}{n-1}$ , (ii)  $\frac{1}{n-1} \leq \frac{1}{l} - \frac{1}{n-1}$ , (iii)  $\frac{1}{n-1} \leq \frac{1}{q+1} + \frac{1}{n-1}$ , (iv)  $\frac{1}{k} - \frac{1}{n-1} \leq \frac{1}{q+1} + \frac{1}{n-1}$ .



Since  $l < k$ , (i) is true. (ii) holds for  $n > 3$  as discussed at the beginning of subcase 1.2  $\frac{1}{l} > \frac{1}{k} > \frac{2}{n-1}$ ; for  $n = 3$ , take  $s = p$  and then  $l = 1$ , so (ii) still holds. (iii) is obvious. (iv) is equivalent to  $\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{n-1}$ , which is guaranteed by (1.2).

So, we put (3.18) and (3.19) in (3.17) and get

$$\begin{aligned} G_2(R) \leq & CR^{n+2}(\|D_x^2 u\|_{l+\varepsilon}(R) + R^{-1}\|D_x u\|_1(R) + R^{-2}\|u\|_1(R))^{\tau_1} \\ & \times (\|D_x^2 u\|_k(R) + R^{-1}\|D_x u\|_1(R) + R^{-2}\|u\|_1(R))^{1-\tau_1} \\ & \times (\|D_x^2 v\|_{1+\varepsilon}(R) + R^{-1}\|D_x v\|_1(R) + R^{-2}\|v\|_1(R))^{\tau_2} \\ & \times (\|D_x^2 v\|_m(R) + R^{-1}\|D_x v\|_1(R) + R^{-2}\|v\|_1(R))^{1-\tau_2}. \end{aligned} \quad (3.25)$$

Then by Proposition 2.10, there exists  $\tilde{R} \in [R, 2R]$  such that

$$\begin{aligned} G_2(\tilde{R}) & \leq CR^{n+2} R^{-\frac{p\beta\tau_1}{1+\varepsilon/l}} \left( (R^{-n}F(4R))^{\frac{1}{k}} + R^{-\frac{\alpha+2}{1+\varepsilon}} + R^{-\alpha-2} \right)^{1-\tau_1} \\ & \times R^{-\frac{q\alpha\tau_2}{1+\varepsilon/l}} \left( (R^{-n}F(4R))^{\frac{1}{m}} + R^{-\frac{\beta+2}{1+\varepsilon}} + R^{-\beta-2} \right)^{1-\tau_2} \\ & \leq CR^{-a_2^\varepsilon} F^{1-\delta_2}(4R), \end{aligned}$$

where the last inequality is due to  $R^{-\frac{n}{k}} > R^{-\alpha-2}$  and  $R^{-\frac{n}{m}} > R^{-\beta-2}$ . Meanwhile,

$$a_2 = a_2^\varepsilon = -n - 2 + \frac{p\beta\tau_1}{1+\varepsilon/l} + \frac{q\alpha\tau_2}{1+\varepsilon/l} + n\frac{1-\tau_1}{k} + n\frac{1-\tau_2}{m}, \quad (3.26)$$

$$1 - \delta_2 = \frac{1-\tau_1}{k} + \frac{1-\tau_2}{m}. \quad (3.27)$$

Similar to subcase 1.1, we only need to prove

$$a_2^0 > 0, \quad \delta_2 > 0.$$

Surprisingly, similar to  $a_1 \approx (\alpha + \beta + 2 - n)\delta_1$ , we have  $a_2 \approx (\alpha + \beta + 2 - n)\delta_2$  since we can prove  $a_2^0 = (\alpha + \beta + 2 - n)\delta_2$ . Indeed,

$$\begin{aligned} a_2^0 & = -n - 2 + p\beta(\tau_1 - 1) + p\beta + q\alpha(\tau_2 - 1) + q\alpha + n(1 - \delta_2) \\ & = -n - 2 - p\beta k \frac{1-\tau_1}{k} - q\alpha m \frac{1-\tau_2}{m} + \alpha + \beta + 4 + n(1 - \delta_2) \\ & = \alpha + \beta + 2 - n - (\alpha + \beta + 2)(1 - \delta_2) + n(1 - \delta_2) \\ & = (\alpha + \beta + 2 - n)\delta_2, \end{aligned}$$

where the third equality above is due to  $p\beta k = (p+1)\beta = (q+1)\alpha = q\alpha m$  and  $(p+1)\beta = \alpha + \beta + 2$ . So, we only need to prove  $\delta_2 > 0$  or equivalently by (3.20), (3.21) and (3.27),

$$(m - \frac{k}{l})\frac{1}{z} + (\frac{k}{n-1} + (m-1)(k-1))\frac{1}{l} + \frac{m-2}{n-1} - (m-1) > 0, \quad (3.28)$$

To achieve this, we take the upper bound of  $\frac{1}{z}$  in (3.24) and see whether (3.28) holds.

**Case 2.1.1** If  $\frac{1}{l} - \frac{1}{n-1} \geq \frac{1}{q+1} + \frac{1}{n-1}$ , then let  $\frac{1}{z} = \frac{1}{q+1} + \frac{1}{n-1}$ , and (3.28) becomes,

$$\begin{aligned} & (\frac{1}{pq} - \frac{p+1}{p(q+1)})\frac{1}{l} + \frac{q+1}{q}(\frac{1}{n-1} + \frac{1}{q+1}) + \frac{1-q}{(n-1)q} - \frac{1}{q} > 0 \\ & \Leftrightarrow (\frac{1}{pq} - \frac{p+1}{p(q+1)})\frac{1}{l} + \frac{2}{q(n-1)} > 0 \\ & \Leftrightarrow -\frac{2}{\beta s} + \frac{2}{n-1} > 0 \\ & \Leftrightarrow s\beta > n-1. \end{aligned}$$

**Case 2.1.2** If  $\frac{1}{l} - \frac{1}{n-1} < \frac{1}{q+1} + \frac{1}{n-1}$ , then let  $\frac{1}{z} = \frac{1}{l} - \frac{1}{n-1}$ , and (3.28) becomes,

$$\begin{aligned}
& (m - \frac{k}{l})(\frac{1}{l} - \frac{1}{n-1}) + (\frac{k}{n-1} + (m-1)(k-1))\frac{1}{l} + \frac{m-2}{n-1} - (m-1) > 0 \\
& \Leftrightarrow -\frac{k}{l^2} + (m + \frac{k}{n-1} + \frac{k}{n-1} + (m-1)(k-1))\frac{1}{l} + \frac{m-2}{n-1} - (m-1) > 0 \\
& \Leftrightarrow -\frac{k}{l^2} + (\frac{p}{p+1} + \frac{1}{q} + \frac{2}{n-1})\frac{k}{l} > \frac{2}{n-1} + \frac{1}{q} \\
& \Leftrightarrow -\frac{k}{l^2} + (1 + k(\frac{2}{n-1} + \frac{1}{q}))\frac{1}{l} - (\frac{2}{n-1} + \frac{1}{q}) > 0 \\
& \Leftrightarrow (\frac{k}{l} - 1)(\frac{1}{l} - (\frac{2}{n-1} + \frac{1}{q})) < 0 \\
& \Leftrightarrow \frac{1}{k} < \frac{1}{l} < \frac{2}{n-1} + \frac{1}{q}.
\end{aligned}$$

Notice that  $\frac{1}{l} < \frac{2}{n-1} + \frac{1}{q}$  holds under the assumption of case 2.1.2, and  $\frac{1}{k} < \frac{1}{l}$  since  $l < k$ . In all, (3.28) always holds under our assumption  $n - s\beta < 1$ .

**Subcase 2.2**  $m \geq n - 1$ .

First, we have for any  $\gamma \in [1, \infty)$ ,

$$W^{1,m}(\mathbb{S}^{n-1}) \hookrightarrow L^\gamma(\mathbb{S}^{n-1}).$$

Then we claim  $\frac{1}{l} > \frac{1}{n-1}$ . Suppose  $\frac{1}{l} \leq \frac{1}{n-1}$ , then  $k > l \geq n - 1$ , hence  $p \leq \frac{1}{n-2}$ , which is not possible due to (3.5). Take  $\frac{1}{z} = \frac{1}{l} - \frac{1}{n-1}$  then

$$W^{1,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^z(\mathbb{S}^{n-1}).$$

Therefore, by Sobolev embedding and (3.17)

$$\begin{aligned}
G_2(R) & \leq CR^n(\|D_x u\|_z(R) + R^{-1}\|u\|_1(R))(\|D_x v\|_{z'}(R) + R^{-1}\|v\|_1(R)) \\
& \leq CR^{n+2}(\|D_x^2 u\|_{l+\varepsilon} + R^{-1}\|D_x u\|_1 + R^{-2}\|u\|_1)(\|D_x^2 v\|_m + R^{-1}\|D_x v\|_1 + R^{-2}\|v\|_1).
\end{aligned}$$

Similarly to previous work, there exists a  $\tilde{R} \in [R, 2R]$  such that

$$\begin{aligned}
G_2(\tilde{R}) & \leq CR^{n+2}R^{\frac{-p\beta}{1+\varepsilon/l}} \left( (R^{-n}F(4R))^{\frac{1}{m}} + R^{-\frac{\beta+2}{1+\varepsilon}} + R^{-\beta-2} \right) \\
& \leq CR^{-a_2^\varepsilon}F^{1-\delta_2}(4R),
\end{aligned}$$

where

$$a_2 = a_2^\varepsilon = -n - 2 + \frac{p\beta}{1+\varepsilon/l} + \frac{n}{m}, \quad (3.29)$$

$$1 - \delta_2 = \frac{1}{m}. \quad (3.30)$$

Direct verification shows that

$$a_2^0 = (\alpha + \beta + 2 - n)\delta_2,$$

and obviously  $\delta_2 > 0$  so  $\alpha_2^0 > 0$ .

Thus, we have proved Case 2.

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